

# Spherically-symmetric solutions with a chain of $n$ internal Ricci-flat spaces

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## Abstract

The Schwarzschild solution is generalized for the case of  $n$  internal Ricci-flat spaces. It is shown that in the four-dimensional section of the metric a horizon exists only when the internal space scale factors are constant. The scalar-vacuum generalization of the solution is also presented. [This paper is the English translation of the part of Chapter. 2.4 of the author's PhD dissertation (Moscow, 1989).]

Here we derive an exact solution to vacuum Einstein equations  $R_{MN} = 0$  in a spherically-symmetrical case when all internal spaces  $M_1, \dots, M_n$  are Ricci-flat [1, 2].

So, the problem is to find a solution for the metric of the form

$$g = - e^{2\gamma(u)} dt \otimes dt + e^{2\alpha(u)} du \otimes du + e^{2\beta_0(u)} d\Omega^2 + \sum_{i=1}^n e^{2\beta_i(u)} g_{(i)} \quad (1)$$

on the manifold

$$M = \mathbb{R} \times \mathbb{R}_* \times S^2 \times M_1 \times \dots \times M_n, \quad (2)$$

obeying the vacuum Einstein eqs., where  $M_i$  is a Ricci-flat manifold of dimension  $N_i$  with the metric  $g_{(i)}$ ,  $i = 1, \dots, n$ ,  $d\Omega^2$  is the canonical metric on  $S^2$ ,  $\mathbb{R}_* \subset \mathbb{R}$  and  $u$  is a radial-type variable connected with  $r$  by the relation  $r = e^{\beta_0(u)}$ . Denote  $\gamma = \beta_{-1}$ ,  $N_{-1} = 1$ ,  $N_0 = 2$ . Let  $\alpha = \alpha_0 \equiv \sum_{\nu=-1}^n \beta_\nu N_\nu$  ( $u$  is a harmonic radial variable).

Then Einstein eqs.  $R_{MN} = 0$  read ( $A' \equiv \frac{d}{du}A$ )

$$\begin{aligned}
\sum_{\nu=-1}^n [-\beta''_{\nu} + \alpha'_0 \beta'_{\nu} - (\beta'_{\nu})^2] N_{\nu} &= 0, \\
\beta''_i &= 0, \quad i = -1, 1, \dots, n, \\
\beta''_0 &= e^{2\alpha_0 - 2\beta_0}.
\end{aligned} \tag{3}$$

Solving the set of equations (3) (here it is convenient to use the variable  $x = \beta_0 - \alpha_0$ ) we get

$$\begin{aligned}
\beta_i &= A_i \bar{u} + D_i, \quad i = -1, 1, \dots, n, \\
\beta_0 &= -\ln f - \sum_{\nu \neq 0} (A_{\nu} \bar{u} + D_{\nu}) N_{\nu}, \\
\alpha_0 &= -2 \ln f - \sum_{\nu \neq 0} (A_{\nu} \bar{u} + D_{\nu}) N_{\nu},
\end{aligned} \tag{4}$$

where

$$f = f(\bar{u}, B) = \begin{cases} \frac{sh(\sqrt{B}\bar{u})}{\sqrt{B}}, & B > 0, \\ \bar{u}, & B = 0. \end{cases} \tag{5}$$

In (4)  $\bar{u} = \varepsilon(u + u_0)$ ,  $\varepsilon = \pm 1$ ;  $u_0, A_i, D_i$  are arbitrary constants,  $i = -1, 1, \dots, n$ .  $B$  is defined by the relation

$$2B = \left( \sum_{\nu \neq 0} A_{\nu} N_{\nu} \right)^2 + \sum_{\nu \neq 0} N_{\nu} A_{\nu}^2 \tag{6}$$

( $\sum_{\nu \neq 0}$  means the summation over  $\nu : \nu = -1, 1, \dots, n$ ). If we re-denote the constants

$$\begin{aligned}
c_i &= e^{2D_i}, \quad a_i \sqrt{B} = -A_i, \quad i = 1, \dots, n; \\
c &= e^{D_{-1}}, \quad a \sqrt{B} = -A_{-1}, \\
L &= 2\sqrt{B} \exp\left(-\sum_{\nu \neq 0} D_{\nu} N_{\nu}\right)
\end{aligned} \tag{7}$$

and introduce a new variable  $R$ :

$$R = e^{-\sum_{\nu \neq 0} D_{\nu} N_{\nu}} \times \begin{cases} \frac{2\sqrt{B}}{1 - e^{-2\pi\sqrt{B}}}, & B > 0, \\ 1/\bar{u}, & B = 0, \end{cases} \tag{8}$$

then the solution (1), (4) reads [1, 2]

$$g = -c^2 dt \otimes dt \left(1 - \frac{L}{R}\right)^a + dR \otimes dR \left(1 - \frac{L}{R}\right)^{-a - \sum_{i=1}^n a_i N_i} + d\Omega^2 R^2 \left(1 - \frac{L}{R}\right)^{1 - a - \sum_{i=1}^n a_i N_i} + \sum_{i=1}^n c_i g_{(i)} \left(1 - \frac{L}{R}\right)^{a_i}, \quad (9)$$

$R > L$ , where constants  $L \geq 0, c, c_1, \dots, c_n > 0$  are arbitrary and  $a, a_1, \dots, a_n$  obey the relation following from (6)

$$\left(a + \sum_{i=1}^n a_i N_i\right)^2 + a^2 + \sum_{i=1}^n a_i^2 N_i = 2. \quad (10)$$

Solution (9), (10) for special case  $n = 1$  was considered earlier in [3, 4]. [For one-dimensional internal spaces see also [5, 6] ( $n = 1$ ) and [7] ( $n = 2, 3$ ).]

When  $L = 0$  the solution (9) is trivial: in this case 4-dimensional part of the metric (9) is flat and scale factors for  $g_{(i)}$  are constant. For  $L > 0$  and

$$a - 1 = a_1 = \dots = a_n = 0 \quad (11)$$

the solution (9) is the sum of the Schwarzschild solution (with the gravitational radius  $L$ ) and the tensor field  $\sum_{i=1}^n c_i g_{(i)}$ . Let  $L > 0$ , then  $a > 0$  corresponds to an attraction and  $a < 0$  describes a repulsion.

Now let us study the problem of a horizon for the solution (9). Consider  $g_4$  which is the 4-dimensional section of the metric (9). For the metric  $g_4$  in the non-trivial case  $L > 0$  the horizon at  $R = L$  takes place only when (11) holds. Indeed, for a radial light geodesic obeying  $ds_4^2 = 0$  we have

$$c(t - t_0) = \int_R^{R_0} dx \left(1 - \frac{L}{R}\right)^{-a - \frac{1}{2} \sum_{i=1}^n a_i N_i}. \quad (12)$$

Relation (10) is equivalent to the following identity

$$\left(a + \frac{1}{2} \sum_{i=1}^n a_i N_i\right)^2 = 1 - \frac{1}{2} \sum_{i=1}^n a_i^2 N_i - \frac{1}{4} \left(\sum_{i=1}^n a_i N_i\right)^2. \quad (13)$$

If not all  $a_i = 0$  ( $i = 1, \dots, n$ ), then due to (13)

$$\left| a + \frac{1}{2} \sum_{i=1}^n a_i N_i \right| < 1, \quad (14)$$

and so the integral (12) is convergent for  $R = L$ , i.e. a radial light ray reaches the surface  $R = L$  at a finite time. If  $a_1 = \dots = a_n = 0$  then due to (10)  $a = \pm 1$ . When  $a = 1$ ,  $a_1 = \dots = a_n = 0$  the metric  $g_4$  coincides with the Schwarzschild solution having a horizon at  $R = L$ . If  $a = -1$ ,  $a_1 = \dots = a_n = 0$  then the integral (12) is finite for  $R = L$  and hence the horizon is absent. Thus, for the metric  $g_4$  (which is the 4-dimensional section of the metric (9)) the surface  $R = L$  is a horizon only in the trivial case (11) when scale factors of internal spaces are constant and 4-section of the total metric coincides with the Schwarzschild solution.

Solution (9) could be easily generalized when a minimally coupled scalar field is taken into account. In this case the action of the model

$$S = \frac{1}{2} \int d^D z \, |g|^{1/2} \left( \frac{R[g]}{\kappa^2} - g^{MN} \partial_M \varphi \partial_N \varphi \right) \quad (15)$$

leads to equations of motion

$$R_{MN} = \kappa^2 \partial_M \varphi \partial_N \varphi, \quad (16)$$

$$\Delta \varphi = 0, \quad (17)$$

where  $\Delta$  is the Laplace operator for the metric  $g$ . For the metric (9) and the scalar field  $\varphi = \varphi(u)$  (where  $u$  is a harmonic radial variable) eq. (17) reads:  $\varphi'' = 0$ , or, equivalently,

$$\varphi = Qu + \bar{\varphi}_0, \quad (18)$$

where  $Q$  and  $\bar{\varphi}_0$  are constants. The substitution of the metric (1) and the scalar field from (18) into eqs. (16) leads us to a set of equations which differs from (3) by the presence of the term  $\kappa^2 Q^2$  in the right hand side of the first equation of the set (3). Solving this modified set of equations (along a line as it was done for the set (3)) we get

$$\varphi = \frac{1}{2} q \ln \left( 1 - \frac{L}{R} \right) + \varphi_0, \quad (19)$$

where  $q$  and  $\varphi_0$  are constants ( $q$  is scalar charge), and the metric  $g$  is given by the same formula (9) but instead of (10) the constants  $a, a_1, \dots, a_n$  obey the relation [2]

$$\left( a + \sum_{i=1}^n a_i N_i \right)^2 + a^2 + \sum_{i=1}^n a_i^2 N_i + \kappa^2 q^2 = 2. \quad (20)$$

The scalar-vacuum solution for  $n = 1$  was considered earlier in [8]. It is easy to prove using (20) that a horizon for  $R = L > 0$  takes place only when

$$q = a - 1 = a_1 = \dots = a_n = 0. \quad (21)$$

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